MONOMIAL SOLUTIONS TO GENERALIZED YANG-BAXTER EQUATIONS IN LOW DIMENSIONS

An Undergraduate Research Scholars Thesis

by

ANDREW SCHMIDT NEMEC

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Research Advisor: Dr. Eric C. Rowell

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ABSTRACT

Monomial Solutions to Generalized Yang-Baxter Equations in Low Dimensions. (May 2015)

Andrew Schmidt Nemec
Department of Computer Science and Engineering
Department of Mathematics
Texas A&M University

Research Advisor: Dr. Dr. Eric C. Rowell
Department of Mathematics

Unitary solutions to the Yang-Baxter equation are important to quantum information science because they lead to unitary representations of the braid group, which can be used to design quantum logic gates that make up topological quantum circuits. By finding new unitary solutions to the Generalized Yang-Baxter equation in low dimensions and classifying them, we will be able to find new representations of the braid group which may lead to new designs for quantum logic gates used in quantum computers. Because it is extremely difficult to find solutions to the Generalized Yang-Baxter equation, we will narrow our search to set-theoretical solutions, that is, solutions that are also permutaiton matrices.

NOMENCLATURE

YBE	Yang-Baxter equation
gYBE	Generalized Yang-Baxter Equation
$\otimes$	Kronecker Product
$\oplus$	Bitwise XOR
S_n	Symmetric group on n letters

CHAPTER I

INTRODUCTION

Yang-Baxter Equations

The Yang-Baxter equation (YBE) was first developed in 1944 by L. Onsanger, when he applied the well known Y-Δ transform (i.e. star-triangle transform) from the field of electrical network analysis to his work on the Ising model in statistical mechanics. The importance of these equations came to light through the independent efforts of R. A. Baxter and C. N. Yang; in the late 1970s, L. Fadeev coined the term Yang-Baxter to recognize the contributions of these two physicists to the development of the equations. Today the YBE is important not only in the field of statistical mechanics but also in quantum field theory, knot theory, quantum topology, and quantum information science, which is the application that interests us the most [1].

The d-dimensional YBE is a matrix equation for a nondegenerate complex matrix $R = \left(R_{ij}^{kl}\right), i, j, k, l = 1, 2, \ldots, d$. The simplest way to view the YBE is as an equation of linear operators between vector spaces. Let V be a d-dimensional complex vector space with a fixed basis $\{e_i\}, i = 1, 2, \ldots, d$. R defines an invertible operator $R : V \otimes V \to V \otimes V$ by $R\left(e_i \otimes e_j\right) = \sum_{k,l=1}^{d} R_{ij}^{kl} e_k \otimes e_l, i, j = 1, 2, \ldots, d$, where $\{e_a \otimes e_b\}, a, b = 1, 2, \ldots, d$ is a basis for $V \otimes V$. The Yang-Baxter equation is then:

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R), \tag{I.1}$$

where I is the identity of V and $\otimes$ is the Kronecker product; for two matrices $X = (x_{ij})_{m \times n}$ and $Y = (y_{kl})_{p \times q}$,

$$X \otimes Y = \begin{bmatrix} x_{11}Y & \ldots & x_{1n}Y \\ \vdots & \ddots & \vdots \\ x_{m1}Y & \ldots & x_{mn}Y \end{bmatrix}. \tag{I.2}$$

Solutions to the YBE are found by filling R with variables, evaluating I.1, and solving the resulting system of polynomials. This turns out to be a nontrivial task: the d-dimensional YBE produces a set of d^6 cubic polynomials with d^4 variables [1]. In 1993, J. Hietarinta found all of the solutions to the 2-dimensional YBE [5]; the 3-dimensional YBE remains unsolved, although many solutions are known. In 2003, H. Dye classified all of the unitary solutions discovered by Hietarinta, which was significant because in the field of quantum information unitary R-matrices, that is R such that $R^\dagger R = I$, are of special interest, as they lead to unitary representations of the braid group [2].

Generalized Yang-Baxter Equations

In 2007 E. Rowell, Y. Zhang, Y.-S. Yu, and M.-L. Ge introduced the Generalized Yang-Baxter equation (gYBE) [8]. If V is a complex vector space of dimension d, the (d, m, l)-gYBE is an equation for an invertible operator $R : V^{\otimes m} \to V^{\otimes m}$ such that

$$\left(R \otimes I_V^{\otimes l}\right)\left(I_V^{\otimes l} \otimes R\right)\left(R \otimes I_V^{\otimes l}\right) = \left(I_V^{\otimes l} \otimes R\right)\left(R \otimes I_V^{\otimes l}\right)\left(R \otimes I_V^{\otimes l}\right). \tag{I.3}$$

The d-dimensioal YBE corresponds to the $(d, 2, 1)$-gYBE. However, not all solutions to the gYBE are interesting: for example, if R is a solution to the (d, m, l)-gYBE where $l \leq m$, then $R = \lambda I_{V^{\otimes m}}$ for some nonzero complex scalar λ. To our knowledge, all of the currently known solutions to the $(2, 3, 1)$- and $(2, 3, 2)$-gYBEs are given in [1, 4, 7].

One property of the YBE is that all of its solutions lead to braid group representations; however, we must impose the further restriction of far commutativity on our gYBE solutions in order to retain this desirable property.

Set-Theoretical Solutions

Due to the enormous number of equations that must be solved in order to completely solve a YBE, it is perhaps easier to study certain R-matrices whose forms make it easier to check if they are solutions. The simplest set of these R-matrices are the set-theoretical solutions proposed by V. Drinfeld, and further explored by P. Etingof [3]. Let X be a nondegenerate symmetric set, that is a set X, $|X| < \infty$, with an invertible mapping $R : X^2 \to X^2, (i, j) \mapsto (f_i(i, j), g_j(i, j))$ such that R is a nondegenerate unitary solution to the Yang-Baxter equation. Then $R \in Perm(X \times X)$, and there are only $(|X|^2)!$ permutations that must be searched. Letting $V = span\{e_1, \ldots, e_k\}, k = |X|$, then $V \otimes V = span\{e_i \otimes e_j\}$ and our solution is $R(e_i \otimes e_j) = e_{f_i(i,j)} \otimes e_{g_j(i,j)}$.

This approach can be extended to the gYBE. The set-theoretical $(|X|, m, l)$-gYBE is an equation for an $R \in Perm\left(X^m\right)$ such that

$$\left(R \times id_X^l\right)\left(id_X^l \times R\right)\left(R \times id_X^l\right) = \left(id_X^l \times R\right)\left(R \times id_X^l\right)\left(id_X^l \times R\right). \tag{I.4}$$

However, this simply means we need to look for $|X|^m \times |X|^m$ permutation matrices that solve the $(|X|, m, l)$-gYBE. Since a permutation matrix tensored with the identity on either side is also a permutation matrix, and the set of all $n \times n$ permutation matrices form a group $G \cong S_n$, multiplying the three terms on either side of the gYBE together will result in permutation matrices. Verifying whether two permutation matrices are equal is simply comparing them for nonzero elements in the same positions, it is an extremely quick process to determine whether or not a permutation matrix satisfies the gYBE. Also, since there are $n!$ possible $n \times n$ permutation matrices, it is relatively simple to test every permutation matrix of a given size for low dimensions of the gYBE.

CHAPTER II

PERMUTATION SOLUTIONS IN DIMENSION 2

Generating Set-Theoretical Solutions

To find all of the set-theoretical solutions to the $(2,3,1)$- and $(2,3,2)$-gYBE, the most straight forward approach is to generate all $8!$ permutation matrices of order 8 and then substitute to determine if they satisfy either one. To accomplish this, we have implemented Heap's algorithm for generating permutations in $Maple$ so that it treats the columns of the matrix it is operating on as the elements that are permuted. Heap's algorithm was chosen because it operates by exchanging a single pair of elements at a time from the previous permutation, making it one of the most effective algorithms for generating permutations. Even though an itereative implementation of Heap's algorithm is faster, we chose to implement it recursively since it is simpler and easier to modify for permutations of different sizes [9]. All of the set-theoretical solutions are given in Appendix A in permutation notation.

One characteristic of set-theoretical solutions is that they can be generalized into monomial solutions; in order to find all monomial solutions to the gYBE, one has simply to find a set-theoretical solution Q and multiply it by a diagonal matrix A such that each entry of A is an independent variable. Then $R := AQ$ can be plugged into the gYBE and we obtain a set of polynomial equations describing the necessary restrictions on the variables such that R is a solution. This holds true because of Lemma 1:

Lemma 1 *Suppose R is an invertible monomial matrix that satisfies the gYBE. Then the permutation matrix obtained by replacing all nonzero entries of R with 1 is also a solution to the gYBE.*

Proof of Lemma 1

There are two cases: (1) for a given column of R the nonzero entries on either side of the gYBE are in different rows, or (2) they are in the same row. In (1), we have $abc = 0$ for some $a, b, c \in \mathbb{C}$ which are values from R. However, $abc = 0 \implies$ either a, b, or $c = 0$; this means that at least one of R's columns is all zeros, meaning R is not invertible, violating our initial assumptions therefore (1) can never occur. For (2), we have $abc - def = 0$ for $a, b, c, d, e, f \in \mathbb{C}$ which are variables from R. It is easy to see that setting all of the vatiables to 1 will always solve the equation therefore the permutation matrix with nonzero entries in the same locations as R is also a solution to the gYBE. ■

Restrictions on Monomial Matrices

To find the restrictions on the variables for the monomial matrices, we used a Gröbner basis algorithm in $Maple$ and then factored to find which variables needed to be set equal each other. We then replaced those variable with an equivalent expression and repeated the process. In addition, the $(2, 3, 1)$-gYBE solutions need to satisfy the far commutativity relation:

$$\left(R \otimes I_2^{\otimes 2}\right)\left(I_2^{\otimes 2} \otimes R\right) = \left(I_2^{\otimes 2} \otimes R\right)\left(R \otimes I_2^{\otimes 2}\right). \tag{II.1}$$

The complete set of monomial solutions to the $(2, 3, 1)$- and $(2, 3, 2)$-gYBE are given in Appendix B.

CHAPTER III

SOLUTION CLASSIFICATION

After implementing the procedures outlined in the previous chapter in a $Maple$ program, we ran it and came up with 15 solutions satisfying the (2,3,1)-gYBE and far commutativity, and 11 solutions satisfying the (2,3,2)-gYBE, which may be found in Appendix A. We then ran them through our procedure to find what restrictions must be put on variables as described in the previous chapter so that the resulting monomial matrix also satisfies the gYBE, and the results of this can be found in Appendix B. Unfortunately, few of these solutions are locally conjugate to one another, so classification in this fashion is relatively pointless. Likewise, there seems to be few patterns among the permutation cycles, so classifying them using that approach is futile.

Boolean Representation of Solutions

After much searching, we determined that the best way to organize the solutions is to make use of the inherent structure of the matrices and Boolean algebra. By inherent strucure, we mean that each of the three sets that are tensored together have two values, which we will call 0 and 1. If we choose 0 from all of the sets, we have $|0, 0, 0>$, which we will call ”0” due to this being the binary representation of 0. Likewise, if we take all permutations of these elements, we come up with a total of 8 choices, numbered 0 to 7, each one associated with a different column of the matrix solution. As it turns out, all of the set-theoretic solutions to the (2,3,1)- and (2,3,2)-gYBE can be written in a form involving only a bitwise XOR, represented by $\oplus$, and bitwise negation, represented by a bar over the value being negated.

For example, take solution R_{06}. The value 0 is mapped to 3, the value 1 is mapped to 2, and so on. We want to find functions f, g, h such that $|a, b, c >\rightarrow |f(a, b, c), g(a, b, c), h(a, b, c) >$. Since a, b, c can each only have values of either 0 or 1, this reduces to a boolean logic problem, where there are 8 total inputs. The easiest way to find the equations for f, g, h is to use a truth table and then analyze it to find patterns that match basic boolean operations:

Table III.1
Truth Table for R_{06}

a	b	c	$f(a,b,c)$	$g(a,b,c)$	$h(a,b,c)$
0	0	0	0	1	1
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	0	1
1	0	0	1	1	1
1	0	1	1	1	0
1	1	0	0	0	0
1	1	1	0	0	1

Using our knowledge of Boolean logic, we determine that $f = a \oplus b$, $g = \bar{b}$, and $h = \overline{b \oplus c}$. Going through all of the solutions we discovered, we find that they all can be represented using Boolean logic in the same way, using only XOR and negation. The Boolean representation of each solution may be found in Appendix C.

Analysis of Solutions

Looking at all of the solutions in their Boolean representations, we notice several patterns. First, in the (2,3,1)-gYBE solutions, the values of a and c do not appear in the expressions for h and f respectively. We believe this phenomenon to be a result of the far commutativity requirement that we imposed on these solutions. In other words, the values a, b, c can only slide over one slot from their original location. This means that a and c can reside in both their original location and the center slot, but cannot get into the other's original slot, while b can reside in any of the slots.

For the (2,3,2)-gYBE solutions, we notice that with only the exception of the identity (which will always be a solution), a and c do not appear in their original positions. We attribute this to the extra identity in the gYBE, which forces the outermost bits into moving away either one or two slots. Another pattern that we notice is that the central slot is occupied in all but two cases by a plain b, with the other two being occupied by $a \oplus b \oplus c$ and $\overline{a \oplus b \oplus c}$. Again, we attribute this to the structure of the (2,3,2)-gYBE, which can be best seen by looking at this as a digitial combinatorial logic problem: each logic block has three inputs and outputs, and their are five wires connecting them, but only the center wire goes through all three. Therefore, the center slot of the circuit needs to be one of the above values to keep the entire circuit balanced and equal to its ”flip” version.

Extension to Larger Solutions

Using some of the patterns that we discovered in the previous section, we were able to construct solutions to the (2,4,2)- and (2,4,3)-gYBE. For instance we found two solutions to the (2,4,2)-gYBE,

$$X_{01} : |a, b, c, d >\rightarrow |c, d, a, b >$$

$$X_{02} : |a, b, c, d >\rightarrow |c, b, a, d >$$

and we suspect that all 16 permutations of the negations of each are also solutions, which brings our total to 32. However, these are not especially interesting, as they are also (4,2,1)-gYBE (that is the ordinary YBE in 4 dimensions) solutions, which have already been identified, although perhaps never explicitly given anywhere.

However, we were also able to construct five (2,4,3)-gYBE solutions,

$$Y_{01} : |a, b, c, d >\rightarrow |d, b, c, a >$$

$$Y_{02} : |a, b, c, d >\rightarrow |b \oplus c \oplus d, b, c, a \oplus b \oplus c >$$

$$Y_{03} : |a, b, c, d >\rightarrow |\overline{b \oplus c \oplus d}, b, c, \overline{a \oplus b \oplus c} >$$

$$Y_{04} : |a, b, c, d >\rightarrow |\overline{b \oplus c \oplus d}, b, c, a \oplus b \oplus c >$$

$$Y_{05} : |a, b, c, d >\rightarrow |b \oplus c \oplus d, b, c, \overline{a \oplus b \oplus c} >$$

all of which we suspect have not been discovered. Unfortunately, we suspect that these solutions are not the only (2,4,2)- and (2,4,3)-gYBE solutions, and we cannot say whether or not every solutions to these equations can also be written using the same Boolean expressions. However, we hope that our faster evaluation program written primarily in C that does no matrix operations whatsoever can help us to produce a classification for all (2,4,X)-gYBE solutions in the near future.

CHAPTER IV

OPEN QUESTIONS

Solutions to the gYBE provide a way to find new representations of the braid group, which have applications on several fields, including quantum information science. Here we list some of the open questions that we have that relate to the research presented here.

1. Is it possible to represent all 2 dimensional solutions in the same style as the $(2, 3, 1)$- and $(2, 3, 2)$-gYBE solutions using Boolean operators, and is there a way to extend these to higher dimensions?

2. How many of the solutions presented here, if any, produce new representations of the braid group? We plan to explore this question in the near future using the ideas presented in [6].

3. Are there any interesting quantum circuits that can be realized using the new solutions?

4. Is there a better method for finding all set-theoretical solutions? In fact, we implemented our algorithm in C using permutations instead of matrices, and noticed a significant speedup; however, it was not enough to find larger solutions.

Our ultimate goal is to know whether any of the braid representations of the solutions presented here can be realized by physical systems and be used in the design of topological quantum computers. Such devices promise immense boosts to computing power, for instance, allowing us to factor large numbers in sub-exponential time.

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APPENDIX A

PERMUTATION SOLUTIONS

The solutions shown here are the permutation matrices written as permutations of $0, 1, ..., 7$, where 0 corresponds to the first column of the permutation matrix, 1 corresponds to the second column of the matrix, etc.

(2,3,1)-gYBE Solutions

$$R_{00} = () \tag{A.1}$$

$$R_{01} = (0,5)\,(1,4) \tag{A.2}$$

$$R_{02} = (2,4)\,(3,5) \tag{A.3}$$

$$R_{03} = (1,2)\,(5,6) \tag{A.4}$$

$$R_{04} = (1,3)\,(4,6) \tag{A.5}$$

$$R_{05} = (0,6)\,(1,7) \tag{A.6}$$

$$R_{06} = (0,3,5,6)\,(1,2,4,7) \tag{A.7}$$

$$R_{07} = (2,7)\,(3,6) \tag{A.8}$$

$$R_{08} = (0,4,6,2)\,(1,5,7,3) \tag{A.9}$$

$$R_{09} = (0,3)\,(4,7) \tag{A.10}$$

$$R_{10} = (0,6,5,3)\,(1,7,4,2) \tag{A.11}$$

$$R_{11} = (0,2)\,(5,7) \tag{A.12}$$

$$R_{12} = (0,2,6,4)\,(1,3,7,5) \tag{A.13}$$

$$R_{13} = (0,2,3,1)\,(4,6,7,5) \tag{A.14}$$

$$R_{14} = (0,1,3,2)\,(4,5,7,6) \tag{A.15}$$

(2,3,2)-gYBE Solutions

$$S_{00} = () \tag{A.16}$$

$$S_{01} = (1, 4)\,(3, 6) \tag{A.17}$$

$$S_{02} = (0, 5)\,(3, 6) \tag{A.18}$$

$$S_{03} = (1, 6)\,(3, 4) \tag{A.19}$$

$$S_{04} = (0, 7)\,(2, 5) \tag{A.20}$$

$$S_{05} = (1, 4)\,(2, 7) \tag{A.21}$$

$$S_{06} = (0, 5)\,(2, 7) \tag{A.22}$$

$$S_{07} = (0, 4, 5, 1)\,(2, 6, 7, 3) \tag{A.23}$$

$$S_{08} = (0, 1, 5, 4)\,(2, 6, 7, 3) \tag{A.24}$$

$$S_{09} = (0, 4, 5, 1)\,(2, 3, 7, 6) \tag{A.25}$$

$$S_{10} = (0, 1, 5, 4)\,(2, 3, 7, 6) \tag{A.26}$$

APPENDIX B

MATRIX SOLUTIONS

(2,3,1)-gYBE Solutions

$$R_{00} = \begin{bmatrix} \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \end{bmatrix}, |\alpha| = 1 \tag{B.1}$$

$$R_{01} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \beta^2\gamma^{-1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = 1 \tag{B.2}$$

$$R_{02} = \begin{bmatrix} \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot \\ \cdot & \cdot & \gamma & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \gamma & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = 1 \tag{B.3}$$

$$R_{03} = \begin{bmatrix} \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = 1 \tag{B.4}$$

$$R_{04} = \begin{bmatrix} \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta\epsilon\gamma^{-1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = |\epsilon| = 1 \tag{B.5}$$

$$R_{05} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot \\ \gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = 1 \tag{B.6}$$

$$R_{06} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \sqrt{\alpha\beta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{\alpha\beta} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{\alpha\beta} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{\alpha\beta} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.7}$$

$$R_{07} = \begin{bmatrix} \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta^2\gamma^{-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \gamma & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = 1 \tag{B.8}$$

$$R_{08} = \begin{bmatrix} \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.9}$$

$$R_{09} = \begin{bmatrix} \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \\ \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = 1 \tag{B.10}$$

$$R_{10} = \begin{bmatrix} \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \\ \beta^2\alpha^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \beta^2\alpha^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.11}$$

$$R_{11} = \begin{bmatrix} \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \gamma\delta\alpha^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \delta & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = 1 \tag{B.12}$$

$$R_{12} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.13}$$

$$R_{13} = \begin{bmatrix} \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.14}$$

$$R_{14} = \begin{bmatrix} \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.15}$$

(2,3,2)-gYBE Solutions

$$S_{00} = \begin{bmatrix} \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \end{bmatrix}, |\alpha| = 1 \tag{B.16}$$

$$S_{01} = \begin{bmatrix} \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot \\ \cdot & \epsilon\gamma\beta^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \delta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = |\epsilon| = 1 \tag{B.17}$$

$$S_{02} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot \\ \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \delta\gamma\beta^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = 1 \tag{B.18}$$

$$S_{03} = \begin{bmatrix} \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \\ \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \delta & \cdot & \cdot \\ \cdot & \epsilon\gamma\beta^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \delta \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = |\epsilon| = 1 \tag{B.19}$$

$$S_{04} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta \\ \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \cdot & \cdot & \delta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \delta\gamma\beta^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = 1 \tag{B.20}$$

$$S_{05} = \begin{bmatrix} \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \delta\gamma\beta^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \delta & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = 1 \tag{B.21}$$

$$S_{06} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot \\ \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \delta\gamma\beta^{-1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \delta & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = |\gamma| = |\delta| = 1 \tag{B.22}$$

$$S_{07} = \begin{bmatrix} \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.23}$$

$$S_{08} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.24}$$

$$S_{09} = \begin{bmatrix} \cdot & \alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.25}$$

$$S_{10} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot \\ \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot \\ \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha \\ \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, |\alpha| = |\beta| = 1 \tag{B.26}$$

APPENDIX C

LOGIC

(2,3,1)-gYBE Solutions

$$R_{00} : |a, b, c> \rightarrow |a, b, c> \tag{C.1}$$

$$R_{01} : |a, b, c> \rightarrow |\overline{a \oplus b}, b, \overline{b \oplus c}> \tag{C.2}$$

$$R_{02} : |a, b, c> \rightarrow |b, a, c> \tag{C.3}$$

$$R_{03} : |a, b, c> \rightarrow |a, c, b> \tag{C.4}$$

$$R_{04} : |a, b, c> \rightarrow |a, a \oplus b \oplus c, c> \tag{C.5}$$

$$R_{05} : |a, b, c> \rightarrow |\overline{b}, \overline{a}, c> \tag{C.6}$$

$$R_{06} : |a, b, c> \rightarrow |a \oplus b, \overline{b}, \overline{b \oplus c}> \tag{C.7}$$

$$R_{07} : |a, b, c> \rightarrow |a \oplus b, b, b \oplus c> \tag{C.8}$$

$$R_{08} : |a, b, c> \rightarrow |\overline{b}, a, c> \tag{C.9}$$

$$R_{09} : |a, b, c> \rightarrow |a, \overline{c}, \overline{b}> \tag{C.10}$$

$$R_{10} : |a, b, c> \rightarrow |\overline{a \oplus b}, \overline{b}, b \oplus c> \tag{C.11}$$

$$R_{11} : |a, b, c> \rightarrow |a, \overline{a \oplus b \oplus c}, c> \tag{C.12}$$

$$R_{12} : |a, b, c> \rightarrow |b, \overline{a}, c> \tag{C.13}$$

$$R_{13} : |a, b, c> \rightarrow |a, \overline{c}, b> \tag{C.14}$$

$$R_{14} : |a, b, c> \rightarrow |a, c, \overline{b}> \tag{C.15}$$

(2,3,2)-gYBE Solutions

$$S_{00} : |a, b, c >\rightarrow |a, b, c > \tag{C.16}$$

$$S_{01} : |a, b, c >\rightarrow |c, b, a > \tag{C.17}$$

$$S_{02} : |a, b, c >\rightarrow |\overline{b \oplus c}, b, \overline{a \oplus b} > \tag{C.18}$$

$$S_{03} : |a, b, c >\rightarrow |c, a \oplus b \oplus c, a > \tag{C.19}$$

$$S_{04} : |a, b, c >\rightarrow |\overline{c}, \overline{a \oplus b \oplus c}, \overline{a} > \tag{C.20}$$

$$S_{05} : |a, b, c >\rightarrow |b \oplus c, b, a \oplus b > \tag{C.21}$$

$$S_{06} : |a, b, c >\rightarrow |\overline{c}, b, \overline{a} > \tag{C.22}$$

$$S_{07} : |a, b, c >\rightarrow |\overline{c}, b, a > \tag{C.23}$$

$$S_{08} : |a, b, c >\rightarrow |b \oplus c, b, \overline{a \oplus b} > \tag{C.24}$$

$$S_{09} : |a, b, c >\rightarrow |\overline{b \oplus c}, b, a \oplus b > \tag{C.25}$$

$$S_{10} : |a, b, c >\rightarrow |c, b, \overline{a} > \tag{C.26}$$